Taking Bounded Variables Seriously: Extended Beta Binomial, Asymmetric Logit, and Time Series *

Kentaro Fukumoto †

November 17, 2004

Abstract

Bounded variables are so common in the field of political science, though the conventional OLS model is flawed. This paper argues that the extended beta binomial, not normal, distribution is more appropriate and, in addition, shows heterogeneity of unobserved individuals, if sample size is known as it usually is. Also, instead of linear models, I propose the asymmetric logit link for considerting lopsided vulnerability of people to social change. Monte Carlo simulation verifies my model.

`(This is work in progress. Comments are really welcome. Notes for Gov. 3009 readers are included in this font.)`

* This is a paper prepared for the Research Workshop in Applied Statistics (Gov. 3009), Harvard University.

† Visiting Scholar, Reischauer Institute of Japanese Studies, Harvard University, Cambridge, MA, and Associate Professor, Department of Political Science, Gakushuin University, Tokyo, `fukumoto@fas.harvard.edu`.

1 Introduction

Bounded variables are so common in the field of political science.[1] There are two kinds: binary variable's average and the others. The former is average of variables which takes values of either 0 or 1. To put it differently, binary variable's average expresses the rate of units (e.g. individuals) which choose one of two options (yes or no). Examples are; rate of approval of the executive chief and support of parties at survey, share of votes in elections and seats in the legislature, unemployment rate, rate of those who connect to internet, watch campaign ads programs on TV and so on.

The other kind of bounded variables include proportion of specific policies such as welfare and military or debt to the total size of budget (probably not dispersed), average test scores of school or area (overdispersed), rate of covered area of utilities such as sewage or telephone.

Most political scientists implicitly assume that these bounded variables follow the normal distribution and use the OLS estimators. There are, however, some problems. At first, in the case of binary variable's average, the variance of the normal distribution is restricted and not homoskedastic. Estimation without taking them into account might be wrong. The other kind of bounded variables do not have to the normal distribution. Second, relationship between an explained bounded varibale and covariates should not be linear. Otherwise, extreme values of explaining variables would predict values outside boundaries.

This paper addresses these two flaws. If we utilize rough information of sample size, we can improve precision of estimation. And you had better use logit link to bounded variables as well as binary dependent variables. Beyond these, I show how we can exploit more about bounded variables. By assuming the extended beta binomial (EBB) distribution, we can estimate how heterogeneous unobserved inidividuals are in each observation. Also, I propose

[1] A random variable Y is a bounded variable when it has a finite lower bound l and upper bound u. Let $p \equiv f(y) = (y - l)/(u - l)$, then $p \in [0, 1]$, inclusive, and f has the inverse. Therefore, we only have to consider proportion data p in order to study bounded variables in general.

the asymmetric logit link, which enables us to consier how asymmetrically vulnerablity (or resistance) of peoples' response to social change is distributed.

The remainder of this paper is composed as follows. The next section explains the data generation process of bounded variable in terms of stochastic component and systematic one. It also encompasses time series models. In the third section, Monte Carlo simulation shows superiority of the EBB model with asymmetic logit link. Finally, I concluide.

2 Data Generation Process

2.1 Stochastic Component and Samples' Heterogeneity

In this section, I concentrate on binary variables' average only, because beta regression has already been developed for the other type of bounded variable (Ferrari and Cribari-Neto, 2004).

Suppose that there are T observations. Unit of observation may be time (year, quarter, or month), space (local community or country) and so on. An obseravtion at t have n_t individuals. Individual (or sample) i returns binary response $Y_{it} \in \{0, 1\}$ where $i \in \{1, ..., n_t\}$ and $t \in \{1, ..., T\}$. For example, it takes 1 if person i in year t approve the current government and 0 otherwise. Y_{it} is a Bernoulli random variable conditional on π_{it}, that is, $Y_{it} \sim f_{Berboulli}(y_{it}|\pi_{it}) = (\pi_{it})^{y_{it}}(1 - \pi_{it})^{1-y_{it}}$, where $0 < \pi_{it} < 1$. We never observe y_{it} itself, but the sample mean $\bar{y}_t = (1/n_t)\Sigma_{i=1}^{n_t} y_{it}$ like an approval rate of n_t samples in year t.

Here, π_{it} varies not only over observation t but also across individual i. For example, since Republicans tend to approve the Republican President, their π_{it} should be greater than Democrats. Also, a random variable Π_{it} may be dependent on each other. When you live in highly Democratic neighborhood, you are more likely to be persuaded to disapprove the Republican administration. We can not observe π_{it}s themselves. Let $E(\Pi_{it})_t \equiv \mu_t$, however distributed Π_{it} is.

Then,

$$
\begin{aligned}
Y_{it} &\sim f_Y(y_{it}|\mu_t) \\
&= \int f_{Berboulli}(y_{it}|\pi_{it}) f_\pi(\pi_{it}|\mu_t) d\pi_{it} \\
Pr(Y_{it}=1) &= \int \pi_{it} f_\pi(\pi_{it}|\mu_t) d\pi_{it} \\
&= \mu_t \\
Pr(Y_{it}=0) &= \int (1-\pi_{it}) f_\pi(\pi_{it}|\mu_t) d\pi_{it} \\
&= (1-\mu_t) \\
E(Y_{it})_t &= 1 \times Pr(Y_{it}=1) + 0 \times Pr(Y_{it}=0) \\
&= \mu_t \\
V(Y_{it})_t &= E({Y_{it}}^2)_t - E(Y_{it})_t^2 \\
&= 1 \times Pr(Y_{it}=1) + 0 \times Pr(Y_{it}=0) - E(Y_{it})_t^2 \\
&= \mu_t(1-\mu_t)
\end{aligned}
$$

2.1.1 When n_t Is Totally Unknown: Heteroskedastic Normal Distribution

According to the Central Limit Theorem,

$$\bar{Y}_t \overset{a}{\sim} f_{normal}(\bar{y}_t|E(Y_{it})_t, V(Y_{it})_t/n_t) = f_{normal}(\bar{y}_t|\mu_t, \mu_t(1-\mu_t)/n_t)$$

This is heteroskedastic normal distribution. When n_ts are totally unknown, it is impossible to estimate not only μ_ts but also Tn_ts. One of usual ways to avoid this problem is to restrict n_t so that $\mu_t(1-\mu_t)/n_t$ is constant, say, σ^2. For example, when scholars use the OLS model with normal distribution(and actually many do), they implicitly follow this tactics.[2] But this

[2] In this case, we can caluculate $\hat{n_t}$ from $\hat{\mu_t}$ and $\hat{\sigma^2}$, though they are not usually quantities of interest for political scientists.

restriction does not seem plausible in most situations we face. For example, in poll surveys, n_t does not vary so much, but μ_t does. Thus, $\mu_t(1-\mu_t)/n_t$ is probably far from constant.

2.1.2 When Distribution of N_t Is Known: Joint Density

In practice, researchers have some information about N_t such as mean, even if they may not know the exact values of n_ts. If they can assume distribution of N_t and know all its parameters, they will improve their estimation. Let $N_t \sim f_N(n_t)$. Then,

$$\bar{Y}_t \overset{a}{\sim} \int f_{normal}(\bar{y}_t|\mu_t, \mu_t(1-\mu_t)/n_t) f_N(n_t) dn_t$$

Utilizing available infromation about N_t makes estimation of μ_t more precise. `(Note: I bet the natural conjugate prior, inverse gamma distribution, is mathematically convenient, though I do not study this well.)`

2.1.3 When n_t Is Exactly Known: The Extended Beta Binomial (EBB) Distribution

In fact, it is not unusual situation that we know the exact values of n_t. For example, survey reports usually tell sample size. Or we can usually know the number of voters. In this case, we observe $\Sigma_{i=1}^{n_t} y_{it} = n_t \bar{y}_t$, which should be a non negative integer. Then, we can learn how heterogeneous each respondents in the sample are.[3]

[3] Ordinarily, rounded, not exact, $\bar{y}_t$ is reported. For example, only up to the fisrt decimal place of percentage are reported as $\bar{y}_{t,report}$. Denote not rounded true $\bar{y}_t$ as $\bar{y}_{t,true}$. Then,

$$\bar{y}_{t,report} - 0.0005 < \bar{y}_{t,true} \leq \bar{y}_{t,report} + 0.0005$$
$$f_{ebb}(n_t \bar{y}_{t,report}) = \Sigma_{i=\text{round}(n_t \bar{y}_{t,report} - 0.0005)}^{\text{round}(n_t \bar{y}_{t,report} + 0.0005)} f_{ebb}(i)$$

Using Taylor series expansion around $\bar{y}_{t,report}$, this value is approximated by $f_{ebb}(n_t \bar{y}_{t,report}) \times \text{round}(0.001 n_t)$. When, like survey, n_t does not vary so much and round(0.001 n_t) is constant, we do not have to care about this rounding problem.

Here, I assume Π_{it} follows beta distribution in observation t.

$$\begin{aligned}\Pi_{it} &\sim f_{beta}(\pi_{it}|\mu_t,\gamma_t)\\ &= \frac{\Gamma(\mu_t\gamma_t^{-1}+(1-\mu_t)\gamma_t^{-1})}{\Gamma(\mu_t\gamma_t^{-1})\Gamma[(1-\mu_t)\gamma_t^{-1}]}\pi_{it}^{\mu_t\gamma_t^{-1}-1}(1-\pi_{it})^{(1-\mu_t)\gamma_t^{-1}-1}\end{aligned}$$

where $\gamma_t > 0$. If γ_t and therefore its variance, $\mu_t(1-\mu_t)(1+\gamma_t^{-1})^{-1}$, is small enough, π_{it} is probably as large as μ_t. This means that most respondents have similar attitudes. By contrast, as γ_t and the variance becomes larger, π_{it} is more likely to be either near 0 or 1 amd people get polarized.

Then, $N_t\bar{Y}_t$ has the extended beta binomial (EBB) distribution.

$$\begin{aligned}N_t\bar{Y}_t &\sim f_{ebb}(n_t\bar{y}_t|\mu_t,\gamma_t)\\ &= \frac{n_t!}{n_t\bar{y}_t!(n_t(1-\bar{y}_t))!}\Pi_{j=0}^{n_t\bar{y}_t-1}(\mu_t+\gamma_t j)\Pi_{j=0}^{n_t(1-\bar{y}_t)-1}((1-\mu_t)+\gamma_t j)/\Pi_{j=0}^{n_t-1}(1+\gamma_t j)\end{aligned}$$

This is not an asymptotic distribution as the previous two cases are, but the exact one. `(Note: Is expectation` μ_t`? How much is variance?)` Very few articles in political science apply the EBB distribution for bounded dependent variables. `(Note: if you happen to know disproof, I really hope you let me know that.)`[4]

2.2 Systematic Component and Asymmetric Vulnerability

2.2.1 Linear Combination

In studying bounded variables including not only binary response's average but also the other type, most works (unintentionally) assume μ_t is a linear combination of covariates, $\mu_t = x_t\beta$. This value can be below 0 or above 1, though it should not be. For, $\bar{Y}_t$ and

[4] Greene (2000, 4th ed.: 836) implicitly shows that proportions data have binomial distirbution, though he does not consider heterogeneity of π_{it}. King(1989: 119-21) argues that grouped correlated binary variables follow the EBB distribution. In fact, bounded variables can be expressed as grouped binary variables *if and only if* the size of groups (n_t) are found.

therefore μ_t range only between 0 and 1 (inclusive for $\bar{Y}_t$, [0, 1], and exclusive for μ_t, (0, 1)). Therefore, many precedent works are flawed in this regard, too.

2.2.2 (Ordinary) Logit

One of usual links for bounded dependent variable is logit, $0 < \mu_t = (1 + e^{-x_t\beta})^{-1} < 1$. Logit is preferable in another sense that the effect of covariates' change becomes smaller when a depent variable approaches its upper or lower bounds (0 or 1). For exapmle, 1% point unemployment change affects an approval rate more when the rate is 50% than when it is 40% or 60%. This is reasonable. Why? Let me assume that 45% of citizens have disposition to stand by government (which I call Loyalists), another 45% of them are apt to oppose it (Oppositions), and the rest of 10%, Independents, are neutral. They are likely to approve the government in the order of Loyalists, Independents, and Oppositions. When an approval rate of the president is 50% , all 45% Oppositions have already disapproved him (or her?). Hence, 1% point employment loss disappoints, say, 4% points additional Independents. If an approval rate is 40% , all 45% Oppositions and 10% Independents have already said no. Thus, the same 1% point unemployment rise prevents, for example, only 2% points Loyalist from supporting the government, because they have already committed to the government due to some other reasons such as security or religious issues. Similarly, as an approval rate is 60% , 1% point unemployment decrease can persuades approval from, e.g., just 2% points Oppositions. To sum up, Independents are more vulnerable to covariates' change than Loyalists or Oppositions. In general, there are more categories of citizens whose vulnerablity is smaller as they take more extreme values of other covariates. Therefore, S curve of logit function is desirable.

2.2.3 Asymmetric Logit

But symmetricity of logit S curve is not appropaite. Continue the previous example. How about the case where there are 35% Loyalists, 45% Oppositions and 20% Independents? When 40% citizens approve the government, 1% point unemployment increase dampen 4% points Independents' favor, not 2% points Loyalists' one. The effect of covariates' change, or vulnerability of people to the change, is not symmetric over the dependent variable's value of 50%. In general, it is not guaranteed that respondents' vulnerability is symmetric. Probit link does not solve this problem, either.

Therefore, we should use asymmetric link function. While there are several possibilities, due to mathematical consideration, I propose the following *asymmetric logit link.* [5]

$$\mu_t \equiv g(x_t|\beta, \alpha) = (1 + e^{-x_t\beta})^{-\alpha} \qquad \text{where} \quad \alpha > 0$$

α is a symmetricity parameter. In Figure 1, asymmetric logit function curves are depicted for a few values of α. When $x_t\beta$ is $\log\alpha$, the effect of a covariate x_{jt}, $\left|\frac{\partial\mu_t}{\partial x_{jt}}\right| = \left|\beta_j\alpha\mu_t(1 + e^{x_t\beta})^{-1}\right|$, is maximized (or the slope of a graph is steepest) and μ_t is $\mu_t^* \equiv (1 + \alpha^{-1})^{-\alpha}$, which is an inflection point (shown as dots in Figure 1). When α is 1, this reduces to (ordinary) logit and μ_t^* is 0.5. If $\alpha < 1$, μ_t^* is larger than 0.5. By contrast, when $\alpha > 1$, μ_t^* is smaller than 0.5. In addition, if $\alpha\mu_t(1 - \mu_t^{\alpha^{-1}}) > \mu_t(1 - \mu_t)$, ordinary logit overevaluates the effects of covariates. If this inequality is reveresed, it underevaluates. Therefore, when asymmetric logit is appropriate, ordinary logit would result

[5]Maddala (1983: 27-32) and Aldrich and Nelson(1984: 31-5) list several models. The most useful assymetric model is Burr's (1942). According to Aldrich and Nelson (1984: 87), it is $f(Z) = 1 - (1 + Z)^C$ where $Z = x\beta > 0$ or $log(Z) = log(x)\beta Z, X > 0$. They give up this, however, because "C turns out to be difficult to estimate with any precision". Now we can easily estimate C by maximum likelihood. Maddala (1983: 32) write $F(\beta' x) = 1 - (1 + (\beta' x)^c)^k$ where $c, k, \beta' x > 0$, while he complains this is "messier than in the logit model." Yes, it is, though it is more flexible. But, the worst problem of the Burr model is that it constrains $x_t\beta$ so that it is positive, which is inapproproate.

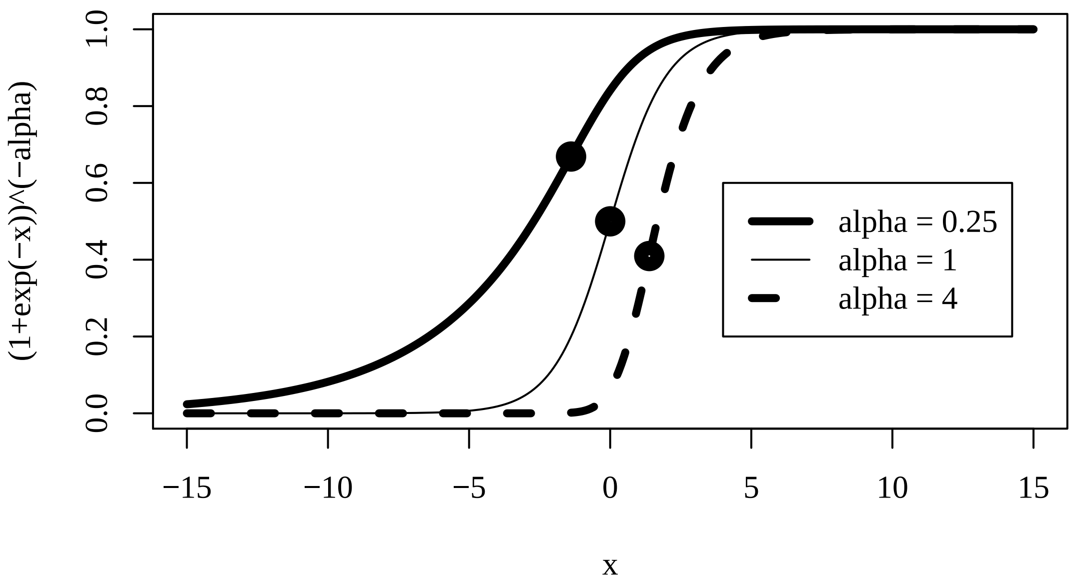

Figure 1: Asymmetric Logit Function Curves

in biased estimations.`(Note: if` $\alpha = 1$ `(logit),` $g(k) = g(-k)$ `for any` k`. Thus,` $-k$ `is conjugate value of` k `on` g`. Then, how about` $\alpha \neq 1$ `? If you can solve this analytically, could you teach me that?)`

If range of $\bar{Y}_t$ is very small, this asymmetric logit curve may be approximated by a first order Taylor series expansion. But this situation is not attractive enough to study and not so common in political science. Hence, a naive linear combination of covariates are not advisable in terms of this, too.

In addition, we can model a dispersion parameter γ_t in the EBB distribution as $\gamma_t = e^{z_t \eta}$ since $\gamma_t > 0$.

2.3 Time Series

Since many bounded variables data are time series, I derive several time series models using asymmetric logit link. I rewrite systematic component as $g^{-1}(\mu_t) = x_t\beta$, because this enables us to express time series models in the conventinoal way. $g^{-1}(\mu_t) = -\log(\mu_t^{-\alpha} - 1)$ is log-powered-odds of μ_t. For example, AR(p) model is

$$g^{-1}(\mu_t) = x_t\beta + \Sigma_{j=1}^{p} g^{-1}(\bar{y}_{t-j})$$

Why? Let $g^{-1}(\bar{y}_t) - g^{-1}(\mu_t) \equiv \xi_t$, then[6]

$$g^{-1}(\bar{y}_t) = x_t\beta + \Sigma_{j=1}^{p} g^{-1}(\bar{y}_{t-j}) + \xi_t$$

Similarly, MA(q) and I(1) is

$$g^{-1}(\mu_t) = x_t\beta + \Sigma_{j=1}^{q} \xi_{t-j}$$
$$g^{-1}(\mu_t) - g^{-1}(\mu_{t-1}) = (x_t - x_{t-1})\beta$$

respectively. Some argue that bounded varibales can not be integrated series (I(1)), because, by construction, their variance can not be more than (quarter of) its range (i.e., 1), though I(1)'s variance should explode as time passes (Beck). But it is not appropriate to give up studying whether bounded variables have other characterisitics of unit root process only because of limitation of their range. One of interesting features of integrated series is that they have long memory, that is, it accumulates all past shocks. For bounded variables, we can model this as $g^{-1}(\bar{y}_t) = \Sigma_{j=0}^{\infty} \xi_{t-j}$. Then, $g^{-1}(\bar{y}_t)$ has long memory, its variance explodes as t becomes large, and therefore it is integrated series, though $\bar{y}_t$ is not. Hence, once

[6] Let $\bar{y}_t - \mu_t \equiv \varepsilon_t$, then $E(\varepsilon_t) = 0$. Generally, for non linear link $g, E(g^{-1}(y)) \neq g^{-1}(E(y))$. Hence, $E(\xi_t) \neq 0$. If this model explains time dependence completely, $\mathrm{Cov}(\xi_t, \xi_u) = 0$ for $t \neq u$. Greene (2000, 4th ed.: 835) shows that around the point $\xi_t = 0, E(\xi_t) = 0$ and $V(\xi_t) = g(x_t\beta)(1 - g(x_t\beta))/(n_t g'(x_t\beta)^2)$.

transoformed through (asymetric) logit link, bounded variables also can be integtared series.

Let lag operator $B(x_t) \equiv x_{t-1}$ and polinomial lag operators $\Phi(B^p) = \Sigma_{i=0}^{p} \phi_i B^i$ and $\Theta(B^q) = \Sigma_{j=0}^{q} \theta_j B^j$. Then, ARIMA(p, d, q) is

$$(1-B)^d g^{-1}(\bar{y}_t) = \Phi^{-1}(B^p)\Theta(B^q)\xi_{t-j}$$

In general, a transfer function is

$$(1-B)^d g^{-1}(\bar{y}_t) = (1-B)_k^d \Sigma_{k=1}^{K} \Phi_k^{-1}(B^p)\Theta_k(B^q)(x_{kt}\beta_k) + \Phi_0^{-1}(B^p)\Theta_0(B^q)\xi_t$$

An Error correction model is

$$(1-B)g^{-1}(\bar{y}_t) = (1-B)\Sigma_{k=1}^{K} \Theta_k(B^q)(x_{kt}\beta_k) + \delta(g^{-1}(\bar{y}_t) - (x_{kt}\beta_k)) + \Theta_0(B^q)\xi_t$$

where δ is a discount parameter. I will use this in the section 4.

3 Simulation

In the previous section, I argue for the EBB distribution model with asymmetric logit link. But is it really better than others? I compare each method by Monte Carlo simulation. Parameters are set as follows: $\beta = (-1, 1)$, $\eta = 1$, $\alpha = e^{-1}$. In each simulation, 120 observations are created. In every obseravation t, n_t is generated following negative binomial distribution with mean of 100 and variance of 1100. $x_t = (1, x_{1,t})$ and $x_{1,t}$ has N(0, 1). z_t is always 1. According to this setting and the data generation process described in the previous section, $\bar{y}_t$ is randomly chosen. Given $\bar{y}_t$, $x_{1,t}$ and n_t, I estimate parameters of each model. [7] 50 simluations are repeated. Average mean and variance of point estimates and standard

[7] I use maximum likelihood estimation. To get starting values in optimization for β, I regress log (not powered) odds of $\bar{y}_t$ on x_t. Those for η, α and σ^2 are 1. I calculate these estimates by using R 1.9.1.

	Constant	x1	Log(alpha)	Ancillary
Mean of Point Estimates				
EBB	-0.977	0.999	-0.975	-9.882
restricted	-1.034	1.018	-0.997	
normal	-1.069	1.033	-1.014	-6.086
Standard Deviance of Point Estimates				
EBB	0.480	0.132	0.259	2.532
restricted	0.561	0.161	0.290	
normal	0.582	0.171	0.301	0.153
Mean of Estimated Standard Error				
EBB	0.529	0.144	0.283	11.312
restricted	0.521	0.144	0.272	
normal	0.634	0.188	0.330	0.129
Standard Deviance of Estimated Standard Error				
EBB	0.099	0.043	0.051	20.066
restricted	0.120	0.054	0.053	
normal	0.154	0.0725	0.067	0.000

Table 1: Comparison of Different Distribution Models' Estimators with Asymmetric Logit Link

errors are reported in Table 1.

Since the EBB distribution model with asymmetric logit link is the true model, it is a matter of course that its estimates are close to the true values. What matters is how efficient the estimators of this model are compared with other assumed distributions and links.

First, always using asymmetric logit link, I compare the EBB model with the normal distribution model. Also, I consider the case where we know N_t is distributed at its first quartile with 25% of probability, at its median with 50% and at its third quartile with 25%. I call this "restricted normal". For all parameters, actual standard deviances of point estimates are the lowest in the EBB model. On average, those of the normal distribution model are 22% larger. As for coefficient of contant terms, the latter model might not reject the null hypothesis mistakenly, while the former would. Hence, different methods can bring about different conclusions. The restricted normal distribution model improves the ordinary normal model, because the number of parameters to be estimated is smaller by one. But it is still 17% less efficient than the EBB model. As a result, means of point estimates are the closest to the true values in the EBB model, too. In addition, the EBB model enables us to know how heterogeneous samples are by η (ancillary in Table 1). The normal model estimates $\sigma^2 = \mu_t(1-\mu_t)/n_t$ (ancillary in Table 1), which makes no sense in political science.

When it comes to averages of estimated standard error, the EBB model have 18% smaller values than the normal model. The restricted normal model performs as well as the EBB model. Besides, the EBB model have the least standard deviance of standard errors, which is 34% less than those of the normal model. From the above, I conclude that, when an explained variable is bounded, the EBB estimators are appropriate and more efficient than the normal model.

Second, I study differences among asymmetric logit, ordinary logit and linear links. It is meaningless to compare coefficients of different links. Therefore, I look at log likelihoods (Table 2). When the EBB distribution is employed, asymmetric logit produces higher likelihood than ordinary logit.[8] And in the case of the normal distribution, too, likelihood of asymmetric logit is larger than those of ordinary logit and linear links. Therefore, if the true systematic component is asymmetric logit, which is not unusual in the social system, estimates by way of other links are less probable.

`(Note: alternativbely, how should I compare performance of each method?)`

[8]The linear link model with the EBB distribution is incomputable, because linear combination can be below 0 or above 1.

Link	Distribution	Log Likelihood
Asymmetric Logit	EBB	-7388.3
Ordinary Logit	EBB	-7393.0
Asymmetric Logit	normal	195.8
Ordinary Logit	normal	192.6
Linear	normal	191.9

Table 2: Comparison of Log Likelihood of Different Models

4 An Empirical Application: The Approval Rate of the Japanese Cabinets

(Note: not yet completed)

5 Conclusion

Bounded variables are more common than political scientists are aware of. In spite of this ubiquitousness, due attention has not been paid to their idyosyncratic features. Taking them seriously, this paper not only addresses shortcomings of the conventional methods but also enhances our ability to study bounded variables.

My argument is that the EBB distribution reflects the exact data generation process and lets us know how heteregeneous inidividuals respond, even if we can not observe them. Even if sample size is unknown, information of its distribution and parameters makes us narrow the confidence interval of estmation.

I also propose the asymetric logit link instead of linear or ordinary logit links. This link guarantees that predicted values are within the boundaries and relaxes the symmetricity of ordinary logit curve. By way of this, we can model unbalanced vulnerability of respondents to covariate changes.

Monte Carlo simulation shows that the normal distritbution model's estimators are less efficient than the EBB model. Besides, asymmetric logit link makes estimators more probable. These results imply that all precedent works which study bounded dependent variables might have wrong conclusion. My model is a simple enough module to incorporate into any other model. I hope this helps future research of bounded variabels in political science as well as any (social) science.

6 Bibliography

(Note: not yet completed. I think the asymetric logit link is new, though I am not yet sure. If you happen toknow related works, it would be appreciated if you teach me that. Also, I am looking for (political science) articles which apply the EBB model *for proportion data*.)